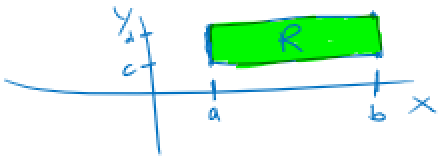
MATH 130A Review: Fubini Theorem

Facts to Know

Fubini's theorem

$$\iint_R f(x,y)\,dA = \int_{y=c}^{y=d} \underbrace{\int_{x=a}^{x=b} f(x,y)\,dx}\,dy = \int_{x=a}^{x=b} \underbrace{\int_{y=c}^{y=d} f(x,y)\,dy}\,dx$$

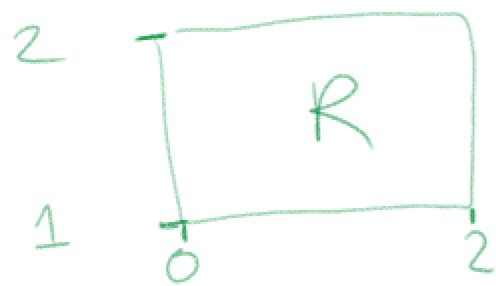
where $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$.



Examples

1. Calculate $\iint_R (y + xy^{-2})\,dA$ where $R = \{(x,y) : 0 \le x \le 2,\ 1 \le y \le 2\}$

$$\int_{y=1}^{y=2}\int_{x=0}^{x=2}\left(y + xy^{-2}\right)dx\,dy$$



$$= \int_1^2 dy \left[yx + y^{-2}\left(\tfrac{1}{2}x^2\right) \right|_0^2$$

$$= \int_1^2 dy \left(2y + 2y^{-2} \right)$$

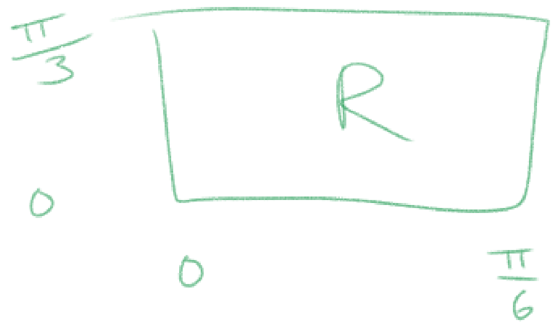
$$= \left[y^2 + 2(-1)y^{-1} \right|_1^2$$

$$= 2^2 + 2(-1)2^{-1} - \left(1^2 + 2(-1)1^{-1}\right)$$

$$= 4$$

2. Calculate $\iint_R x\sin(x+y)\,dA$ where $R = \{(x,y) : 0 \le x \le \frac{\pi}{6},\, 0 \le y \le \frac{\pi}{3}\}$

$$\int_{x=0}^{x=\frac{\pi}{6}} \int_{y=0}^{y=\frac{\pi}{3}} x \sin(x+y)\, dy\, dx$$



$$= \int_0^{\frac{\pi}{6}} dx \left[-x\cos(x+y) \Big|_0^{\pi/3} \right.$$

$$= \int_0^{\pi/6} dx \left(-x\cos\left(x+\frac{\pi}{3}\right) + x\cos(x) \right)$$

$$= \int_0^{\pi/6} \underbrace{x}_{u} \overbrace{\left(\cos(x) - \cos\left(x+\frac{\pi}{3}\right)\right) dx}^{dv}$$

$$= x\left(\sin(x) - \sin\left(x+\frac{\pi}{3}\right)\right)\Big|_0^{\pi/6} - \int_0^{\frac{\pi}{6}} \left(\sin(x) - \sin\left(x+\frac{\pi}{3}\right)\right) dx$$

$$= \underbrace{\frac{\pi}{6}\left(\frac{1}{2} - 1\right)}_{-\frac{\pi}{12}} \underbrace{- \left[-\cos(x) + \cos\left(x+\frac{\pi}{3}\right) \Big|_0^{\pi/6} \right.}_{+\frac{\sqrt{3}}{2} + \left(-1 + \frac{1}{2}\right)} = -\frac{\pi}{12} + \frac{\sqrt{3}}{2} - \frac{1}{2}$$

3. Calculate $\iint_R y\sin(xy)\,dA$ where $R = \{(x,y) : 1 \le x \le 2,\, 0 \le y \le \pi\}$

$$\int_0^{\pi} \int_1^{2} y\sin(xy)\, dx\, dy$$

$$= \int_0^{\pi} dy \left[-\cos(xy) \Big|_1^2 \right.$$

$$= \int_0^{\pi} \left(-\cos(2y) + \cos(y) \right) dy$$

$$= 0$$